

COHOMOLOGICAL INVARIANTS OF ALGEBRAIC GROUPS AND THE MORAVA K -THEORY

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ABSTRACT. In the present article we discuss different approaches to cohomological invariants of algebraic groups over a field. We focus on the Tits algebras and on the Rost invariant and relate them to the Morava K -theory. Furthermore, we discuss oriented cohomology theories of affine varieties and the Rost motives for different cohomology theories in the sense of Levine–Morel.

1. INTRODUCTION

The notion of a Tits algebra was introduced by Jacques Tits in his celebrated paper on irreducible representations [Ti71]. This invariant of a linear algebraic group G plays a crucial role in the computation of the K -theory of twisted flag varieties by Panin [Pa94] and in the index reduction formulas by Merkurjev, Panin and Wadsworth [MPW96]. It has important applications to the classification of linear algebraic groups and to the study of associated homogeneous varieties.

Tits algebras are examples of cohomological invariants of algebraic groups of degree 2. The idea to use cohomological invariants in the classification of algebraic groups goes back to Jean-Pierre Serre. In particular, Serre conjectured the existence of an invariant of degree 3 for groups of type F_4 and E_8 . This invariant was later constructed by Markus Rost for all G -torsors, where G is a simple simply-connected algebraic group, and is now called the Rost invariant (see [GMS03]).

Furthermore, the Milnor conjecture (proven by Voevodsky) provides a classification of quadratic forms over fields in terms of the Galois cohomology, i.e., in terms of cohomological invariants.

In the present article we discuss different approaches to Tits algebras and generalize some of them to invariants of higher degree. In particular, we consider the Morava K -theories, which are the universal oriented cohomology theories in the sense of Levine–Morel with respect to the Lubin–Tate formal group law (see Section 6). It turns out that the Morava K -theories (more precisely the Morava-motives) detect, whether certain cohomological invariants of an algebraic group are zero or not. In particular, the second Morava K -theory is responsible for the Rost invariant (see Theorem 6.16). We remark that in the same spirit Panin showed in [Pa94] that the Grothendieck’s K^0 functor detects the triviality of the Tits algebras.

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Besides that, we discuss another approach to cohomological invariants which uses an exact sequence of Voevodsky (4.5) (see below). For example, this sequence was used in [Sem13] to construct an invariant of degree 5 modulo 2 for some groups of type E_8 and to solve a problem posed by Serre. Besides this, we discuss oriented cohomology theories of affine varieties and the Rost motives for different cohomology theories in the sense of Levine–Morel.

We try to keep the exposition elementary and try to avoid technical generalizations, which would rather hide the ideas. For example, we mostly assume that the groups under considerations are of inner type, though this can be avoided introducing more notation in the formulae.

It is quite amazing that different ideas from algebra, geometry and topology come together, when dealing with cohomological invariants of algebraic groups. For example, the Tits algebras are related to representation theory, K -theory, but also to motivic cohomology of simplicial schemes. Looking at the invariants of higher degree one finds relations with algebraic cobordism of Levine–Morel, classifying spaces of algebraic groups (Totaro, Morel, Voevodsky) and motives; see e.g. [Me13], [MNZ13], [SV14].

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2. DEFINITIONS AND NOTATION

In the present article we assume that F is a field of characteristic 0. This assumption is not needed at the most places. In some places it can be removed by changing the étale topology by the fpqc topology. Nevertheless, we would like to avoid a too technical exposition. Aside from that, at some places (e.g. when we consider the Morava K -theory) the assumption on the characteristic is needed. By F_{sep} we denote a separable closure of F .

Let G be a semisimple linear algebraic group over a field F (see [Spr], [KMRT], [GMS03]). A G -torsor over F is an algebraic variety P equipped with an action of G such that $P(F_{\text{sep}}) \neq \emptyset$ and the action of $G(F_{\text{sep}})$ on $P(F_{\text{sep}})$ is simply transitive.

The set of isomorphism classes of G -torsors over F is a pointed set (with the base point given by the trivial G -torsor G) is in natural one-to-one correspondence with the (non-abelian) Galois cohomology set $H_{\text{et}}^1(F, G)$.

Let A be some algebraic structure over F (e.g. an algebra or quadratic space) such that $\text{Aut}(A)$ is an algebraic group over F . Then an algebraic structure B is called a *twisted form* of A , if over a separable closure of F the structures A and B are isomorphic. There is a natural bijection between $H_{\text{et}}^1(F, \text{Aut}(A))$ and the set of isomorphism classes of the twisted forms of A .

For example, if A is an octonion algebra over F , then $\text{Aut}(A)$ is a group of type G_2 and $H_{\text{et}}^1(F, \text{Aut}(A))$ is in 1-to-1 correspondence with the twisted forms of A , i.e., with the octonion algebras over F (since any two octonion algebras over F are isomorphic over a

separable closure of F and since any algebra, which is isomorphic to an octonion algebra over a separable closure of F , is an octonion algebra).

By $\mathbb{Q}/\mathbb{Z}(n)$ we denote the Galois-module $\varinjlim \mu_l^{\otimes n}$ taken over all l (see [KMRT, p. 431]).

In the article we use notions from the theory of quadratic forms over fields (e.g. Pfister-forms, Witt-ring). We follow [KMRT], [Lam], and [EKM]. Further, we use the notion of motives; see [Ma68], [EKM].

3. ALGEBRAIC CONSTRUCTIONS OF TITS ALGEBRAS

In this section we will briefly describe two classical constructions of Tits algebras following Tits [Ti71].

3.1 (Construction using the representation theory of algebraic groups). Let G_0 be a split semisimple algebraic group of rank n over F and T be a split maximal torus of G_0 . Denote by \widehat{T} the group of characters of T . This is a free abelian group of rank n . Then there is a natural one-to-one correspondence between the isomorphism classes of the irreducible finite dimensional representations of G_0 and the elements in $\Lambda_+ \cap \widehat{T}$, where Λ_+ denotes the cone of dominant weights. This correspondence associates with an irreducible representation of G_0 its highest weight. We remind that we have a global assumption that $\text{char } F = 0$. Nevertheless, we remark that this one-to-one correspondence holds over fields of arbitrary characteristic, but for a given highest weight the dimension of the respective irreducible representation depends on the characteristic of the field.

Let now G be an arbitrary (not necessarily split) semisimple algebraic group over F which is a twisted form of G_0 . A Tits algebra of G corresponding to an element $\omega \in \Lambda_+ \cap \widehat{T}$ is a central simple algebra A over F such that there exists a group homomorphism $\rho: G \rightarrow \text{GL}_1(A)$ such that the representation $\rho \otimes F_{\text{sep}}: G \otimes F_{\text{sep}} \rightarrow \text{GL}_1(A \otimes_F F_{\text{sep}})$ of the split group $G \otimes F_{\text{sep}}$ is the representation with the highest weight ω .

Let Γ denote the absolute Galois group of F . Tits showed that for any Γ -invariant ω the Tits algebra exists and is unique up to an isomorphism. Moreover, there exists a group homomorphism (called the Tits homomorphism)

$$\begin{aligned} \beta: (\widehat{T}/\Lambda_r)^\Gamma &\rightarrow \text{Br}(F) \\ \lambda &\mapsto [A_\lambda] \end{aligned}$$

where Λ_r is the root lattice and A_λ is the Tits algebra of G of the weight $\chi(\lambda)$, where $\chi(\lambda) \in \Lambda_+ \cap \widehat{T}$ is a unique representative of λ in the coset \widehat{T}/Λ_r . If G is a group of inner type, then the action of Γ on \widehat{T}/Λ_r is trivial.

Example 3.2. 1). If $G = \text{SL}_1(A)$, where A is a central simple algebra of degree $n+1$, then the Tits algebra of the first fundamental weight ω_1 (corresponding to the standard representation of SL_{n+1}) is the algebra A itself. The Tits algebra for the last fundamental weight ω_n (corresponding to the dual representation) is the opposite algebra A^{op} .

2). If $G = \text{Spin}_{2n+1}(q)$, where q is a regular quadratic form of dimension $2n+1$, then the Tits algebra of the weight ω_n (enumeration of simple roots follows Bourbaki) is the even Clifford algebra of q . This corresponds to the spinor representation. If $n > 1$, then the Tits algebra of G for the standard representation with the highest weight ω_1 is the split matrix algebra of degree $2n+1$.

3). If G is a group of inner type E_6 (resp. of type E_7), then the Tits algebra of the weight ω_1 (resp. ω_7) is a central simple algebra of degree 27, index dividing 27 and exponent dividing 3 (resp. of degree 56, index dividing 8 and exponent dividing 2).

4). The Tits algebras of groups of types G_2 , F_4 and E_8 are split matrix algebras.

3.3 (Construction using the boundary homomorphism). Another classical construction of Tits algebras goes as follows.

Let G be a split semisimple algebraic group over F with center $Z = Z(G)$. The short exact sequence

$$1 \rightarrow Z \rightarrow G \rightarrow G/Z \rightarrow 1$$

of algebraic groups induces the long exact sequence of cohomology

$$H_{\text{et}}^1(F, G) \rightarrow H_{\text{et}}^1(F, G/Z) \xrightarrow{\partial} H_{\text{et}}^2(F, Z).$$

We remark that G/Z is the adjoint group of the same type as G .

For an irreducible representation $\rho: G \rightarrow \text{GL}_n$ denote by λ_ρ the restriction of ρ to the center Z . Then λ_ρ is a homomorphism from Z to $Z(\text{GL}_n) = \mathbb{G}_m$, i.e., $\lambda_\rho \in \text{Hom}(Z, \mathbb{G}_m) = \widehat{T}/\Lambda_r$.

Consider the composite map

$$H_{\text{et}}^1(F, G/Z) \xrightarrow{\partial} H_{\text{et}}^2(F, Z) \xrightarrow{(\lambda_\rho)_*} H_{\text{et}}^2(F, \mathbb{G}_m) = \text{Br}(F).$$

Then an element $\xi \in H_{\text{et}}^1(F, G/Z)$ maps under this composite map to the class of the Tits algebra A_ρ of the (inner) twisted group ${}_\xi G$ corresponding to the representation ρ . (Instead of a split G one can start with a quasi-split group G . Then the analogous construction will cover all twisted forms of G .)

Example 3.4. Let $G = \text{SL}_n$ and let ρ be the standard n -dimensional representation of G . Then $Z = \mu_n$, $G/Z = \text{PGL}_n$, and an element $\xi \in H_{\text{et}}^1(F, \text{PGL}_n)$ corresponds to a central simple algebra A over F of degree n . The $H_{\text{et}}^2(F, Z) = {}_n \text{Br}(F) := \{x \in \text{Br}(F) \mid nx = 0\} \subset \text{Br}(F)$ and the boundary homomorphism ∂ maps ξ to the class of A in the Brauer group of F . The composite map $(\lambda_\rho)_* \circ \partial$ maps ξ also to $[A]$, which is the Tits algebra for ρ .

Let now $G = \text{PGL}_n$ and ρ be an irreducible representation of G . Then $Z = 1$ and the Tits algebra of ρ is a split matrix algebra. This corresponds to the fact that the elements of the root lattice Λ_r map to 0 under the Tits homomorphism.

4. GEOMETRIC CONSTRUCTIONS OF TITS ALGEBRAS

4.1 (Tits algebras and the Picard group). Another construction of Tits algebras is related to the Hochschild–Serre spectral sequence. For a smooth variety X over F one has

$$H_{\text{et}}^p(\Gamma, H_{\text{et}}^q(X_{\text{sep}}, \mathcal{F})) \Rightarrow H_{\text{et}}^{p+q}(X, \mathcal{F})$$

where $X_{\text{sep}} = X \times F_{\text{sep}}$ and \mathcal{F} is an étale sheaf. The induced 5-term exact sequence is

$$0 \rightarrow H_{\text{et}}^1(\Gamma, H_{\text{et}}^0(X_{\text{sep}}, \mathcal{F})) \rightarrow H_{\text{et}}^1(X, \mathcal{F}) \rightarrow H_{\text{et}}^0(\Gamma, H_{\text{et}}^1(X_{\text{sep}}, \mathcal{F})) \rightarrow H_{\text{et}}^2(\Gamma, H_{\text{et}}^0(X_{\text{sep}}, \mathcal{F}))$$

Let $\mathcal{F} = \mathbb{G}_m$ and X be a smooth projective variety. Then

$$H_{\text{et}}^1(\Gamma, H_{\text{et}}^0(X_{\text{sep}}, \mathbb{G}_m)) = H_{\text{et}}^1(\Gamma, F_{\text{sep}}^\times) = 0$$

by Hilbert 90, $H_{\text{et}}^1(X, \mathbb{G}_m) = \text{Pic}(X)$, $H_{\text{et}}^0(\Gamma, H_{\text{et}}^1(X_{\text{sep}}, \mathbb{G}_m)) = H_{\text{et}}^0(\Gamma, \text{Pic } X_{\text{sep}}) = (\text{Pic } X_{\text{sep}})^\Gamma$, and $H_{\text{et}}^2(\Gamma, H_{\text{et}}^0(X_{\text{sep}}, \mathbb{G}_m)) = H_{\text{et}}^2(\Gamma, F_{\text{sep}}^\times) = \text{Br}(F)$. Thus, we obtain an exact sequence

$$(4.2) \quad 0 \rightarrow \text{Pic } X \rightarrow (\text{Pic } X_{\text{sep}})^\Gamma \rightarrow \text{Br}(F)$$

The map $\text{Pic } X \rightarrow (\text{Pic } X_{\text{sep}})^\Gamma$ is the restriction map and the homomorphism

$$(\text{Pic } X_{\text{sep}})^\Gamma \xrightarrow{f} \text{Br}(F)$$

was described by Merkurjev and Tignol in [MT95, Section 2] when X is the variety of Borel subgroups of a semisimple algebraic group G . Namely, the Picard group of X_{sep} can be identified with the free abelian group with basis $\omega_1, \dots, \omega_n$ consisting of the fundamental weights. If ω_i is Γ -invariant (e.g. if G is of inner type), then $f(\omega_i) = [A_i]$ is the Tits algebra of G corresponding to the (fundamental) representation with the highest weight ω_i (see [MT95] for a general description of the homomorphism f).

Moreover, one can continue the exact sequence (4.2), namely, the sequence

$$0 \rightarrow \text{Pic } X \rightarrow (\text{Pic } X_{\text{sep}})^\Gamma \rightarrow \text{Br}(F) \rightarrow \text{Br}(F(X))$$

is exact, where the last map is the restriction homomorphism (see [MT95]).

4.3 (Tits algebras and K^0). There is another interpretation of the Tits algebras related to Grothendieck's K^0 functor. Let G be a semisimple algebraic group over F of inner type and X be the variety of Borel subgroups of G . By Panin [Pa94] the K^0 -motive of X is isomorphic to a direct sum of $|W|$ motives, where W denotes the Weyl group of G . Denote these motives by L_w , $w \in W$.

For $w \in W$ consider

$$\rho_w = \sum_{\{\alpha_k \in \Pi \mid w^{-1}(\alpha_k) \in \Phi^-\}} w^{-1}(\omega_k) \in \Lambda,$$

where Π is the set of simple roots and Φ^- is the set of negative roots. Using these elements one can describe the *Steinberg basis* of $K^0(X_K)$ over a splitting field K of G ; see [Pa94, Section 12.5], [QSZ12, Section 2].

Over a splitting field K of G , the motive $(L_w)_K$ is isomorphic to a Tate motive and the restriction homomorphism $K^0(L_w) \rightarrow K^0((L_w)_K) = \mathbb{Z}$ is an injection $\mathbb{Z} \rightarrow \mathbb{Z}$ given by the multiplication by $\text{ind } A_w$, where $[A_w] = \beta(\rho_w) \in \text{Br}(F)$ for the Tits homomorphism β . In particular, different motives L_w can be parametrized by the Tits algebras.

Moreover, if all Tits algebras of G are split, then the K^0 -motive of X is a direct sum of Tate motives over F .

4.4 (Tits algebras and simplicial varieties). Let Y be a smooth irreducible variety over F . Consider the standard simplicial scheme \mathcal{X}_Y associated with Y , i.e. the simplicial scheme

$$Y \rightrightarrows Y \times Y \rightrightarrows Y \times Y \times Y \cdots$$

Then for all $n \geq 2$ there is a long exact sequence of cohomology groups (see [Ro07, Cor. 2.2] and [Vo11, Proof of Lemma 6.5]):

$$(4.5) \quad 0 \rightarrow H_{\mathcal{M}}^{n,n-1}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z}) \xrightarrow{f} H_{\text{et}}^n(F, \mathbb{Q}/\mathbb{Z}(n-1)) \rightarrow H_{\text{et}}^n(F(Y), \mathbb{Q}/\mathbb{Z}(n-1)),$$

where $H_{\mathcal{M}}^{n,n-1}$ is the motivic cohomology and the homomorphism f is induced by the change of topology (from Nisnevich to étale).

Let $n = 2$ and Y be the variety of Borel subgroups of an algebraic group G of inner type. Then $H_{\text{et}}^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br}(F)$ and we have a long exact sequence

$$0 \rightarrow H_{\mathcal{M}}^{2,1}(\mathcal{X}_Y) \xrightarrow{f} \text{Br}(F) \rightarrow \text{Br}(F(Y))$$

Thus, $H_{\mathcal{M}}^{2,1}(\mathcal{X}_Y) = \Lambda/\widehat{T}$, where Λ is the weight lattice, and f turns out to be the Tits homomorphism. This gives one more interpretation of the Tits algebras via a change of topology.

5. HIGHER COHOMOLOGICAL INVARIANTS

The Tits algebras are examples of cohomological invariants of G -torsors of degree 2 (since they lie in $H_{\text{et}}^2(F, \mathbb{G}_m) = \text{Br}(F)$). In general, a cohomological invariant of G -torsors of degree n with values in a Galois-module M is a transformation of functors $H_{\text{et}}^1(-, G) \rightarrow H_{\text{et}}^n(-, M)$ from the category of field extensions of F to the category of pointed sets.

For example, if G is a split orthogonal group of degree n and $M = \mathbb{Z}/2$, then $H_{\text{et}}^1(F, G)$ classifies the isomorphism classes of regular quadratic forms of dimension n over F , $H_{\text{et}}^0(F, M) = \mathbb{Z}/2$, $H_{\text{et}}^1(F, M) = F^\times/F^{\times 2}$ by Hilbert 90, $H_{\text{et}}^2(F, M) = {}_2\text{Br}(F) = \{x \in \text{Br}(F) \mid 2x = 0\}$, and the invariants $H_{\text{et}}^1(F, G) \rightarrow H_{\text{et}}^n(F, M)$ are given for $n = 0, 1, 2$ resp. by the dimension mod 2, by the discriminant, and by the Clifford invariant of the respective quadratic form.

For $n \geq 3$ one can define cohomological invariants for quadratic forms, for which the previous cohomological invariants (of degree strictly less than n) are trivial. Namely, such forms lie in the n -th power of the fundamental ideal I in the Witt ring of F , and there are invariants $e_n: I^n \rightarrow H_{\text{et}}^n(F, \mathbb{Z}/2)$ for all n such that for $q \in I^n$ the invariant $e_n(q) = 0$ iff $q \in I^{n+1}$, $n \geq 0$. Moreover, if q and q' are two quadratic forms, then one can test whether $q \simeq q'$ looking at e_n of the difference $q - q'$ starting from $n = 0$. If $e_n(q - q') \neq 0$, then q and q' are not isomorphic. Otherwise, $q - q' \in I^{n+1}$ and one proceeds to e_{n+1} . Since by the Arason-Pfister Hauptsatz $\bigcap_{n \geq 0} I^n = 0$ (see [Lam, Hauptsatz 5.1, Ch. X]), the invariants e_n allow us to check an isomorphism between two quadratic forms.

Moreover, under some conditions it is possible to reconstruct the original quadratic form from its invariants. Namely, start for simplicity with an even dimensional regular quadratic form q over F . Since $H_{\text{et}}^n(F, \mathbb{Z}/2) \simeq I^n/I^{n+1}$ (see [Lam]), we can choose representatives of elements in $H_{\text{et}}^n(F, \mathbb{Z}/2)$ as linear combinations of n -fold Pfister forms. Computing $e_1(q) \in H_{\text{et}}^1(F, \mathbb{Z}/2)$ we can modify q by $e_1(q)$, which is a linear combination of 1-fold Pfister forms. Since $e_1(q - e_1(q)) = 0$, the form $q - e_1(q) \in I^2$ and we can proceed to the next invariant e_2 . If this process stops, i.e. if starting from some point we will get zeroes (e.g. if the base field has finite cohomological dimension), then saving the representatives of the invariants $e_1(q)$, $e_2(q - e_1(q))$ and so on, will allow us to restore the original form q .

The same philosophy can be applied to other algebraic groups. For example, if G is a simple simply connected algebraic group, then there is an invariant

$$H_{\text{et}}^1(-, G) \rightarrow H_{\text{et}}^3(-, \mathbb{Q}/\mathbb{Z}(2))$$

of degree 3, called the *Rost invariant* (see [GMS03]). If G is the spinor group, this invariant is called the *Arason invariant*.

The Rost invariant can be constructed as follows. Let G be a simple simply-connected algebraic group of inner type and Y be a G -torsor. Then there is a long exact sequence (see [GMS03, Section 9])

$$(5.1) \quad 0 \rightarrow A^1(Y, K_2) \rightarrow A^1(Y_{\text{sep}}, K_2)^\Gamma \xrightarrow{g} \text{Ker} (H_{\text{et}}^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H_{\text{et}}^3(F(Y), \mathbb{Q}/\mathbb{Z}(2))) \rightarrow \text{CH}^2(Y)$$

where the K -cohomology group $A^1(-, K_2)$ is defined in [GMS03, Section 4], Γ is the absolute Galois group, and $Y_{\text{sep}} = Y \times_F F_{\text{sep}}$. Moreover, $A^1(Y_{\text{sep}}, K_2)^\Gamma = \mathbb{Z}$ and $\text{CH}^2(Y) = 0$. The Rost invariant of Y is the image of $1 \in A^1(Y_{\text{sep}}, K_2)$ under the homomorphism g . We remark that sequence (5.1) for the Rost invariant is analogous to the sequence (4.2) for the Tits algebras arising from the Hochschild–Serre spectral sequence.

We remark also that if G is a group of inner type with trivial Tits algebras (simply-connected or not), then there is a well-defined Rost invariant of G itself (not of G -torsors); see [GP07, Section 2].

The idea to use cohomological invariant to study linear algebraic groups and torsors is due to Jean-Pierre Serre. For example, the Serre–Rost conjecture for groups of type F_4 says that the map

$$H_{\text{et}}^1(F, F_4) \hookrightarrow H_{\text{et}}^3(F, \mathbb{Z}/2) \oplus H_{\text{et}}^3(F, \mathbb{Z}/3) \oplus H_{\text{et}}^5(F, \mathbb{Z}/2)$$

induced by the invariants f_3 , g_3 and f_5 described in [KMRT, §40] (f_3 and g_3 are the modulo 2 and modulo 3 components of the Rost invariant), is injective. This allows to exchange the study of the *set* $H_{\text{et}}^1(F, F_4)$ of isomorphism classes of groups of type F_4 over F (equiv. of isomorphism classes of F_4 -torsors or of isomorphism classes of Albert algebras) by the *abelian group* $H_{\text{et}}^3(F, \mathbb{Z}/2) \oplus H_{\text{et}}^3(F, \mathbb{Z}/3) \oplus H_{\text{et}}^5(F, \mathbb{Z}/2)$.

In the same spirit one can formulate the Serre conjecture II, saying in particular that $H_{\text{et}}^1(F, E_8) = 1$ if the field F has cohomological dimension 2. Namely, for such fields $H_{\text{et}}^n(F, M) = 0$ for all $n \geq 3$ and all torsion modules M . In particular, for groups over F there are no invariants of degree ≥ 3 , and the Serre conjecture II predicts that the groups of type E_8 over F themselves are split.

Nowadays there exist a number of techniques to construct and study cohomological invariants. In the literature one can find constructions using K^0 , Chow rings, motivic cohomology. Moreover, the algebraic cobordism, general oriented cohomology theories, classifying spaces of algebraic groups and even motives are useful. For example, in the next section we will describe a relation between invariants and the Morava K -theories.

6. MORAVA K -THEORY

In this section we will introduce a geometric cohomology theory — the Morava K -theory, and prove that it detects the triviality of some cohomological invariants (in particular, of the Rost invariant) of algebraic groups.

Consider the algebraic cobordism Ω of Levine–Morel (see [LM]). By [LM, Thm. 1.2.6] the algebraic cobordism is a universal oriented cohomology theory and there is a (unique)

morphism of theories $\Omega^* \rightarrow A^*$ for any oriented cohomology theory A^* in the sense of Levine–Morel.

Each oriented cohomology theory A is equipped with a 1-dimensional commutative formal group law FGL_A . E.g., for the Chow theory CH^* this is the additive formal group law, for K^0 the multiplicative formal group law and for Ω the universal formal group law. Moreover, these theories are universal for the respective formal group laws.

For a theory A^* we consider the category of A^* -motives with coefficients in a commutative ring R , which is defined in the same way as the category of Grothendieck’s Chow motives with CH^* replaced by $A^* \otimes_{\mathbb{Z}} R$ (see [Ma68], [EKM]). In the present section the ring R is \mathbb{Z} , \mathbb{Q} , or $\mathbb{Z}_{(p)}$ for a prime number p .

For a prime number p and a natural number n we consider the n -th Morava K -theory $K(n)$ with respect to p . Note that we do not include p in the notation. We define this theory as the universal oriented cohomology theory for the Lubin–Tate formal group law of height n with the coefficient ring $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ (see below for the definition of the Lubin–Tate formal group law).

For a variety X over F one has

$$K(n)(X) = \Omega(X) \otimes_{\mathbb{L}} \mathbb{Z}_{(p)}[v_n, v_n^{-1}],$$

and v_n is a ν_n -element in the Lazard ring \mathbb{L} , i.e., an element whose *Milnor number* is not divisible by p^2 (see [Sem13, Section 2] and [LM, Section 4.4.4]). The degree of v_n is negative and equals $-(p^n - 1)$. In particular, $K(n)(\mathrm{Spec} F) = \mathbb{Z}_{(p)}[v_n, v_n^{-1}]$. We remark that usually one considers the Morava K -theory with the coefficient ring $\mathbb{F}_p[v_n, v_n^{-1}]$. For any prime p we define $K(0)$ as $\mathrm{CH} \otimes \mathbb{Q}$.

If $n = 1$ and $p = 2$, one has $K(1)(X) = K^0(X)[v_1, v_1^{-1}] \otimes \mathbb{Z}_{(2)}$, since the Lubin–Tate formal group law is isomorphic to the multiplicative formal group law in this case.

We describe now the formal group law for the n -th Morava K -theory modulo p (the Lubin–Tate formal group law) following [Haz] and [Rav]. The logarithm of the formal group law of the *Brown–Peterson cohomology* equals

$$l(t) = \sum_{i \geq 0} m_i t^{p^i},$$

where $m_0 = 1$ and the remaining variables m_i are related to v_j as follows:

$$m_j = \frac{1}{p} \cdot \left(v_j + \sum_{i=1}^{j-1} m_i v_{j-i}^{p^i} \right).$$

Let $e(t)$ be the compositional inverse of $l(t)$. The Brown–Peterson formal group law is given by $e(l(x) + l(y))$.

The n -th Morava formal group law is obtained from the *BP* formal group law by sending all v_j with $j \neq n$ to zero. Modulo the ideal J generated by p, x^{p^n}, y^{p^n} the formal group law for the n -th Morava K -theory equals

$$\mathrm{FGL}_{K(n)}(x, y) = x + y - v_n \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^{ip^{n-1}} y^{(p-i)p^{n-1}} \mod J.$$

Definition 6.1. For an oriented cohomology theory A and a motive M in the category of A -motives over F we say that M is *split*, if it is a finite direct sum of (twisted) Tate motives over F . Note that this property depends on the theory A . E.g., there exist smooth projective varieties whose motives are split for some oriented cohomology theories, but not for all oriented cohomology theories.

6.2 (Euler characteristic). The Euler characteristic of a smooth projective irreducible variety X with respect to an oriented cohomology theory A^* is defined as the push-forward

$$\pi_*^A(1_X) \in A^*(\mathrm{Spec} F)$$

of the structural morphism $\pi: X \rightarrow \mathrm{Spec} F$. E.g., for $A = K^0[v_1, v_1^{-1}]$ the Euler characteristic of X equals

$$v_1^{\dim X} \cdot \sum (-1)^i \dim H^i(X, \mathcal{O}_X)$$

see [Ful, Ch. 15]. If X is geometrically irreducible and geometrically cellular, then this element equals $v_1^{\dim X}$ (see [Za10, Example 3.6]).

For the Morava K -theory $K(n)$ and a smooth projective irreducible variety X of dimension $d = p^n - 1$ the Euler characteristic modulo p equals the element $v_n \cdot u \cdot s_d$ for some $u \in \mathbb{Z}_{(p)}^\times$, where s_d is the Milnor number of X (see [LM, Sec. 4.4.4]). If $\dim X$ is not divisible by $p^n - 1$, then the Euler characteristic of X equals zero modulo p (see [LM, Prop. 4.4.22]).

6.3 (Rost nilpotence for oriented cohomology theories). Let A be an oriented cohomology theory and consider the category of A -motives over F . Let M be an A -motive over F . We say that the Rost nilpotence principle holds for M , if the kernel of the restriction homomorphism

$$\mathrm{End}(M) \rightarrow \mathrm{End}(M_E)$$

consists of nilpotent correspondences for all field extensions E/F .

Usually Rost nilpotency is formulated for Chow motives. By [CGM05, Sec. 8] it holds for all twisted flag varieties. Note that the proof of [CGM05] works for A -motives of twisted flag varieties for all oriented cohomology theories A in the sense of Levine–Morel satisfying the localization property.

Rost nilpotency is a tool which allows to descent motivic decompositions over E to motivic decompositions over the base field F . E.g., assume that Rost nilpotency holds for M and that we are given a decomposition $M_E \simeq \oplus M_i$ over E into a finite direct sum. The motives M and M_i are defined as a pair (X, ρ) and (X_E, ρ_i) , where X is a smooth projective variety over F , $\rho \in A(X \times X)$ and $\rho_i \in A(X_E \times X_E)$ are some projectors. If we assume further that all ρ_i are defined over F , then $M \simeq \oplus N_i$ for some motives N_i over F , and the scalar extension $(N_i)_E$ is isomorphic to M_i for every i (see [CGM05, Section 8], [PSZ08, Section 2]).

Let R_m denote the (generalized) *Rost motive* of a non-zero pure symbol $\alpha \in H_{\mathrm{et}}^m(F, \mu_p^{\otimes m})$ in the category of Chow motives with $\mathbb{Z}_{(p)}$ -coefficients. By definition R_m is indecomposable and for all field extensions K/F the following conditions are equivalent:

- (1) $(R_m)_K$ is decomposable;
- (2) $(R_m)_K \simeq \bigoplus_{i=0}^{p-1} \mathbb{Z}_{(p)}(b \cdot i)$ with $b = \frac{p^{m-1}-1}{p-1}$;

$$(3) \alpha_K = 0 \in H_{\text{et}}^m(K, \mu_p^{\otimes m}).$$

The fields K from this definition are called splitting fields of R_m .

The Rost motives were constructed by Rost and Voevodsky (see [Ro07], [Vo11]). Namely, for all pure symbols α there exists a smooth projective ν_{m-1} -variety X (depending on α) over F such that the Chow motive of X has a direct summand isomorphic to R_m and for every field extension K/F the motive $(R_m)_K$ is decomposable iff X_K has a 0-cycle of degree coprime to p . Besides that, it follows from Brosnan's extension [Br03, Thm. 3.1] of a result [Ro98, Prop. 1] of Rost that Rost nilpotency holds for R_m . The variety X is called a *norm variety* of α .

E.g., if $p = 2$ and $\alpha = (a_1) \cup \dots \cup (a_m)$ with $a_i \in F^\times$, then one can take for X the projective quadric given by the equation $\langle\langle a_1, \dots, a_{m-1} \rangle\rangle \perp \langle -a_m \rangle = 0$, where $\langle\langle a_1, \dots, a_{m-1} \rangle\rangle$ denotes the Pfister form. (We use the standard notation from the quadratic form theory as in [KMRT] and [EKM].)

By [ViYa07, Sec. 2] there is a unique lift of the Rost motive R_m to the category of Ω -motives and, since Ω is the universal oriented cohomology theory, there is a well-defined Rost motive in the category of A^* -motives for any oriented cohomology theory A^* . We will denote this A -motive by the same letter R_m . By $\mathbb{T}(l)$, $l \geq 0$, we denote the Tate motives in the category of A -motives. If $A = \text{CH} \otimes \mathbb{Z}_{(p)}$, we keep the usual notation $\mathbb{T}(l) = \mathbb{Z}_{(p)}(l)$.

Proposition 6.4. *Let p be a prime number, n and m be natural numbers and $b = \frac{p^{m-1}-1}{p-1}$. For a non-zero pure symbol $\alpha \in H_{\text{et}}^m(F, \mu_p^{\otimes m})$ consider the respective Rost motive R_m . Then*

- (1) *If $n < m - 1$, then the $K(n)$ -motive R_m is a sum of p Tate motives $\bigoplus_{i=0}^{p-1} \mathbb{T}(b \cdot i)$.*
- (2) *If $n = m - 1$, then the $K(n)$ -motive R_m is a sum of the Tate motive \mathbb{T} and an indecomposable motive L such that*

$$K(n)(L) \simeq (\mathbb{Z}^{\oplus(p-1)} \oplus (\mathbb{Z}/p)^{\oplus(m-2)(p-1)}) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}].$$

For a field extension K/F the motive L_K is isomorphic to a direct sum of twisted Tate motives iff it is decomposable and iff the symbol $\alpha_K = 0$.

- (3) *If $n > m - 1$, then the $K(n)$ -motive R_m is indecomposable and its realization is isomorphic to the group $\text{CH}(R_m) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}]$. For a field extension K/F the motive $(R_m)_K$ is decomposable iff $\alpha_K = 0$. In this case $(R_m)_K$ is a sum of p Tate motives.*

Proof. Denote by \overline{R}_m the scalar extension of R_m to its splitting field. By [Ya12, Prop. 11.11] (cf. [ViYa07, Thm. 3.5, Prop. 4.4]) the restriction map for the BP -theory

$$(6.5) \quad \text{res}: BP(R_m) \rightarrow BP(\overline{R}_m) = BP(\text{Spec } F)^{\oplus p}$$

is injective, and the image equals

$$(6.6) \quad BP(R_m) \simeq BP(\text{Spec } F) \oplus I(p, m-2)^{\oplus(p-1)},$$

where $I(p, m-2)$ is the ideal in the ring $BP(\text{Spec } F) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ generated by the elements $\{p, v_1, \dots, v_{m-2}\}$.

- (1) Assume first that $n < m - 1$. Since the ideal $I(p, m-2)$ contains v_n for $n < m - 1$ and v_n is invertible in $K(n)(\text{Spec } F)$, we immediately get that all elements in $K(n)(\overline{R}_m)$

are rational, i.e., are defined over the base field. By the properties of the Rost motives

$$(6.7) \quad \Omega^l(R_m \times R_m) = \bigoplus_{i+j=l} \Omega^i(R_m) \otimes \Omega^j(\overline{R}_m)$$

for all l . Since Ω is a universal theory, the same formula holds for BP and for $K(n)$. Therefore all elements in $K(n)(\overline{R}_m \times \overline{R}_m)$ are rational, and hence by Rost nilpotency for R_m this gives the first statement of the proposition.

(2) Assume now that $n = m - 1$. Let X be a norm variety for the symbol α . In particular, $\dim X = p^{m-1} - 1 = p^n - 1$. Since the Morava–Euler characteristic of X equals $u \cdot v_n$ for some $u \in \mathbb{Z}_{(p)}^\times$ (see Section 6.2), the element $v_n^{-1} \cdot u^{-1}(1 \times 1) \in K(n)(X \times X)$ is a projector defining the Tate motive \mathbb{T} . Thus, we get the decomposition $R_m \simeq \mathbb{T} \oplus L$ for some motive L . We claim that L is indecomposable.

Indeed, by [LM, Thm. 4.4.7] the elements of $K(n)^{p^n-1}(R_m \times R_m)$ are linear combinations of elements of the form $v_n^s \cdot [Y \rightarrow X \times X]$, where Y is a resolution of singularities of a closed subvariety of $X \times X$, and $-s(p^n - 1) + \text{codim } Y = p^n - 1$. In particular, $s = 0, 1, -1$ and $\text{codim } Y = 0, p^n - 1, 2(p^n - 1)$.

By formula (6.7) and by the injectivity of the restriction map for BP , it follows that there are at most three rational projectors in $K(n)(\overline{R}_m \times \overline{R}_m)$ (cf. [Nes14, Section 6 and Proof of Lemma 6.12]). These are the diagonal, the projector $v_n^{-1} \cdot u^{-1}(1 \times 1)$ constructed above and their difference (which defines the motive L). Therefore by Rost nilpotency the motive L is indecomposable over F .

Taking the tensor product $- \otimes_{BP(\text{Spec } F)} K(n)(\text{Spec } F)$ with formula (6.5) and using (6.6) one immediately gets the formula for $K(n)(L)$.

(3) The same arguments show that R_m is indecomposable for the Morava K -theory $K(n)$ for $n > m - 1$. Notice that $\text{CH} = BP/D \cdot BP$, where D is the ideal in $BP(\text{Spec } F)$ generated by v_1, v_2, \dots . Since res in formula (6.5) is injective, we get that

$$\text{CH}(R_m) = \text{Im}(\text{res})/D \cdot \text{Im}(\text{res}).$$

Similarly we obtain a formula for the Morava K -theory $K(n)$, which is obtained from BP by sending all v_i with $i \neq n$ to 0 and localizing in v_n . This formula is the same as for $\text{CH} \otimes_{\mathbb{Z}_{(p)}} [v_n, v_n^{-1}]$ (cf. [Nes14, Example 6.14]). \square

Remark 6.8. This proposition demonstrates a difference between K^0 and the Morava $K(n)$ -theory, when $n > 1$. By [Pa94] K^0 of all twisted flag varieties is \mathbb{Z} -torsion-free. This is not the case for $K(n)$, $n > 1$.

Moreover, the same arguments as in the proof of the proposition show that the connective K -theory $CK(1)$ (see [Cai08]) of Rost motives R_m for $m > 2$ contains non-trivial \mathbb{Z} -torsion.

Remark 6.9. The Chow groups of the Rost motives are known; see [KM02, Thm. 8.1], [KM13, Thm. RM.10], [Ya12, Cor. 10.8], [Vi07, Section 4.1].

The proof of the following proposition is close to [A12, Section 8].

Proposition 6.10. *Let A be an oriented cohomology theory in the sense of Levine–Morel satisfying the localization property. Let Z be a smooth variety over a field F . Assume that*

there exists a smooth projective variety Y with invertible Euler characteristic with respect to A and such that for every point $y \in Y$ (not necessarily closed) the natural pullback

$$A(F(y)) \rightarrow A(Z_{F(y)})$$

is an isomorphism

Then the pullback of the structural morphism $Z \xrightarrow{\pi} \operatorname{Spec} F$ induces an isomorphism

$$A(Z) = A(F).$$

Proof. We have the following localization diagram

$$\begin{array}{ccccccc} \varinjlim_{Y' \subset Y} A(Y') & \longrightarrow & A(Y) & \longrightarrow & A(F(Y)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ \varinjlim_{Y' \subset Y} A(Z \times Y') & \longrightarrow & A(Z \times Y) & \longrightarrow & A(Z_{F(Y)}) & \longrightarrow & 0 \end{array}$$

where the vertical arrows are pullbacks of the respective projections and the limits are taken over all closed subvarieties of Y of codimension ≥ 1 .

By the choice of Y the right vertical arrow is an isomorphism. By induction on dimension of Y the left vertical arrow is surjective. It follows by a diagram chase that the middle vertical arrow is surjective as well.

Let $a: Y \rightarrow \operatorname{Spec} F$ be the structural morphism, $b: Z \times Y \rightarrow Y$ and $c: Z \times Y \rightarrow Z$ be the projections. Consider now another commutative diagram:

$$\begin{array}{ccccc} A(F) & \xrightarrow{\pi^*} & & & A(Z) \\ & \searrow a^* & & \swarrow c^* & \\ & & A(Y) & \xrightarrow{b^*} & A(Z \times Y) \\ & \swarrow a_* & & \searrow c_* & \\ A(F) & \xrightarrow{\pi^*} & & & A(Z) \end{array}$$

\simeq on the left and right vertical arrows

By the above considerations the homomorphism b^* is surjective. The left and the right vertical arrows are isomorphisms, since they are multiplications by the A -Euler characteristic of Y which is invertible.

Therefore by a diagram chase the bottom horizontal arrow is surjective. But $A(F)$ is a direct summand of $A(Z)$. Therefore the bottom arrow is an isomorphism. \square

Let now $(a_1) \cup \dots \cup (a_m) \in H_{\text{et}}^m(F, \mathbb{Z}/2)$ be a pure symbol, $a_i \in F^\times$. The quadratic form $q = \langle\langle a_1, \dots, a_{m-1} \rangle\rangle \perp \langle -a_m \rangle$ is called a *norm form* and the respective projective quadric given by $q = 0$ is called a (projective) *norm quadric*. The respective *affine norm quadric* is an open subvariety of the projective norm quadric given by the equation

$$\langle\langle a_1, \dots, a_{m-1} \rangle\rangle = a_m,$$

i.e. setting the last coordinate to 1.

Corollary 6.11. *Let $0 \leq n < m-1$ and set $p = 2$. Consider the affine norm quadric X^{aff} of dimension $2^{m-1} - 1$ corresponding to a pure symbol in $H_{\text{et}}^m(F, \mathbb{Z}/2)$. Then the pullback of the structural morphism $X^{\text{aff}} \xrightarrow{\pi} \text{Spec } F$ induces an isomorphism*

$$K(n)(X^{\text{aff}}) = K(n)(F).$$

Proof. Let $\alpha := (a_1) \cup \dots \cup (a_m) \in H_{\text{et}}^m(F, \mathbb{Z}/2)$ be our pure symbol, $a_i \in F^\times$, q the norm form for α , and Q the respective projective norm quadric given by $q = 0$. Let Y be the projective norm quadric of dimension $2^n - 1$ corresponding to the subsymbol

$$(a_1) \cup \dots \cup (a_{n+1}) \in H_{\text{et}}^{n+1}(F, \mathbb{Z}/2).$$

We need to check the conditions of Proposition 6.10. By the choice of Y it is a ν_n -variety (see e.g. [Sem13, Section 2]). Therefore by [LM, Prop. 4.4.22] its Morava–Euler characteristic is invertible.

Moreover, the quadratic form q is split completely over $F(y)$ for any point y of Y . In particular, $X_{F(y)}^{\text{aff}}$ is a split odd-dimensional affine quadric. The complement $Q' := Q \setminus X^{\text{aff}}$ is a projective Pfister quadric of dimension $2^{m-1} - 2$, and both Q and Q' are split over $F(y)$. Therefore the Chow motives of Q and Q' over $F(y)$ are direct sums of twisted Tate motives. Moreover, $\text{CH}(X_{F(y)}^{\text{aff}}) = \mathbb{Z}$ (see [Ka01, Theorem A.4] for a more general result).

Denote by $\mathcal{M}(-)$ the Chow motive of a variety $-$. The localization sequence for Chow groups implies that the induced map of $\mathcal{M}(Q') \rightarrow \mathcal{M}(Q)$ over $F(y)$ is surjective in positive codimensions. By [ViYa07, Section 2] the same holds for Ω -motives of Q' and Q over $F(y)$. Therefore $\Omega(X_{F(y)}^{\text{aff}}) = \mathbb{L}$, and, thus, $K(n)(X_{F(y)}^{\text{aff}}) = K(n)(F(y))$. We are done. \square

Let now B be a central simple F -algebra of a prime degree p and $c \in F^\times$. Consider the Merkurjev–Suslin variety

$$\text{MS}(B, c) = \{\alpha \in B \mid \text{Nrd}(\alpha) = c\},$$

where Nrd stands for the reduced norm on B .

Corollary 6.12. *In the above notation the structural morphism induces an isomorphism $A(\text{MS}(B, c)) \simeq A(F)$, when A is Grothendieck’s K^0 or the first Morava K -theory with respect to the prime p .*

Proof. Let $Y = \text{SB}(B)$ denote the Severi–Brauer variety of B . We need to check the conditions of Proposition 6.10. The variety Y is a geometrically cellular ν_1 -variety. Thus, its A -Euler characteristic is invertible.

Over a point $y \in Y$ the variety $\text{MS}(B, c)$ is isomorphic to SL_p , since $\text{MS}(B, c)$ over $F(y)$ is an SL_p -torsor over $F(y)$ and $H_{\text{et}}^1(F(y), \text{SL}_p) = 1$. Since GL_p is an open subvariety in \mathbb{A}^{p^2} , by the localization sequence $\Omega(\text{GL}_p) = \mathbb{L}$. Moreover, GL_p is isomorphic as a variety (not as a group scheme) to $\text{SL}_p \times \mathbb{G}_m$ with the isomorphism sending a matrix α to the pair $(\beta, \det \alpha)$ where β is obtained from α by dividing its first row by $\det \alpha$. The composite morphism

$$\text{SL}_p \hookrightarrow \text{GL}_p \xrightarrow{\simeq} \text{SL}_p \times \mathbb{G}_m \rightarrow \text{SL}_p,$$

where the first morphism is the natural embedding and the last morphism is the projection, is the identity. Taking pullbacks in this sequence, one gets that $\Omega(\text{SL}_p) = \mathbb{L}$ and, hence, $A(\text{SL}_p) = A(F(y))$ for A as in the statement of the present corollary. We are done. \square

Let J be an Albert algebra over F (see [KMRT, Chapter IX]) and N_J denote the cubic norm form on J . For $d \in F^\times$ consider the variety

$$Z = \{\alpha \in J \mid N_J(\alpha) = d\}.$$

The group G of isometries of N_J is a group of type 1E_6 and it acts on Z geometrically transitively.

The proof of the following statement belongs to A. Ananyevskiy.

Corollary 6.13. *In the above notation the natural map $K^0(Z) \rightarrow K^0(F)$ is an isomorphism.*

Proof. Let Y be the variety of Borel subgroups of the group G . We need to check the conditions of Proposition 6.10. The K^0 -Euler characteristic of Y is invertible, since Y is geometrically cellular.

Let $y \in Y$ be a point. Then G splits over $F(y)$, the variety Z has a rational point over $F(y)$ and its stabilizer is the split group of type F_4 , i.e., Z is isomorphic to E_6/F_4 over $F(y)$, where E_6 and F_4 stand for the split groups of the respective Dynkin types.

By the branching rules the restriction homomorphism $R(E_6) \rightarrow R(F_4)$ is surjective, where $R(-)$ denotes the representation ring (see [A12, Section 2]). Therefore Merkurjev's spectral sequence (see [A12, Section 6]) implies that

$$K^0(E_6/F_4) \simeq K^0(F(y)) \otimes_{R(E_6)} R(F_4) \simeq K^0(F(y)).$$

We are done. □

Consider the Witt-ring of the field F and denote by I its fundamental ideal.

Proposition 6.14. *Let $m \geq 1$ and set $p = 2$. A regular even-dimensional quadratic form q belongs to I^m iff the Morava motives $K(n)$ of the respective projective quadric are split for all $0 \leq n < m - 1$.*

Proof. Assume that q does not belong to I^m . Let $1 \leq s < m$ be the maximal number with $q \in I^s$. By [OVV07, Thm. 2.10] there exists a field extension K of F such that q_K as an element of the Witt-ring of F is an anisotropic s -fold Pfister form. By Prop. 6.4 its $(s - 1)$ -st Morava motive is not split. Contradiction.

Conversely, assume that q belongs to I^m and let Q be the respective projective quadric. Then we can present q as a finite sum of (up to proportionality) s -fold Pfister forms with $s \geq m$. We prove our statement using induction on the length of such a presentation in the Witt-ring. If q is an s -fold Pfister-form, then, since $s \geq m > n + 1$, by Prop. 6.4 the $K(n)$ -motive of Q is split.

Let α be an s -fold Pfister form in the decomposition of q . Let X^{aff} be the affine norm quadric of dimension $2^n - 1$ corresponding to a subsymbol of α from $H_{\text{et}}^{n+1}(F, \mathbb{Z}/2)$ (note that $n + 1 < m \leq s$). Then the length of q over $F(X^{\text{aff}})$ is strictly smaller than the length of q over F .

Consider the following commutative diagram of localization sequences:

$$\begin{array}{ccccccc}
\varinjlim_{Y' \subset Q \times Q} K(n)(Y') & \longrightarrow & K(n)(Q \times Q) & \longrightarrow & K(n)(F(Q \times Q)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\varinjlim_{Y' \subset Q \times Q} K(n)(X^{\text{aff}} \times Y') & \longrightarrow & K(n)(X^{\text{aff}} \times Q \times Q) & \longrightarrow & K(n)(X_{F(Q \times Q)}^{\text{aff}}) & \longrightarrow & 0
\end{array}$$

where the vertical arrows are pullbacks of the respective projections and the limits are taken over all closed subvarieties of $Q \times Q$ of codimension ≥ 1 .

The right vertical arrow is an isomorphism by Corollary 6.11. The left vertical arrow is surjective by induction on the dimension of the variety $Q \times Q$ (we do not use here that Q is a quadric). Therefore by a diagram chase the middle vertical arrow is surjective.

But by the localization sequence

$$K(n)(X^{\text{aff}} \times Q \times Q) \rightarrow K(n)((Q \times Q)_{F(X^{\text{aff}})})$$

is surjective. By the induction hypothesis on the length of q , the restriction homomorphism

$$K(n)((Q \times Q)_{F(X^{\text{aff}})}) \rightarrow K(n)((Q \times Q)_K)$$

to a splitting field K of $Q_{F(X^{\text{aff}})}$ is surjective. Therefore the restriction homomorphism

$$K(n)(Q \times Q) \rightarrow K(n)((Q \times Q)_K)$$

to the splitting field K is surjective. In particular, since the projectors for the Morava-motive of Q lie in $K(n)(Q \times Q)$, it follows from Rost nilpotency that the $K(n)$ -motive of Q over F is split. \square

Remark 6.15. The same statement with a similar proof holds for the variety of totally isotropic subspaces of dimension k for all $1 \leq k \leq (\dim q)/2$.

Theorem 6.16. *Let p be a prime number. Let G be a simple algebraic group over F and let X be the variety of Borel subgroups of G . Then*

- (1) *G is of inner type iff the $K(0)$ -motive of X is split.*
- (2) *Assume that G is of inner type. All Tits algebras of G are split iff the K^0 -motive with integral coefficients of X is split.*
- (3) *Assume that G is of inner type and the p -components of the Tits algebras of G are split. Then the p -component of the Rost invariant of G is zero iff the $K(2)$ -motive of X is split.*
- (4) *Let $p = 2$. Assume that G is of type E_8 with trivial Rost invariant. Then G is split by an odd degree field extension iff the $K(m)$ -motive of X is split for some $m \geq 4$ iff the $K(m)$ -motive of X is split for all $m \geq 4$.*

Proof. (1) Since $K(0) = \text{CH} \otimes \mathbb{Q}$ by definition, the statement follows from the fact that G is of inner type iff the absolute Galois group of F acts trivially on $\text{CH}(X_{F_{\text{sep}}}) \otimes \mathbb{Q}$ and from the fact that over a splitting field F_{sep} of G the variety $X_{F_{\text{sep}}}$ is cellular.

(2) Follows from [Pa94]; see also Section 4.3.

(3) First we make several standard reductions. Since all prime numbers coprime to p are invertible in the coefficient ring of the Morava K -theory, by transfer argument we are

free to take finite field extensions of the base field of degree coprime to p . Hence we can assume that not only the p -components of the Tits algebras are split, but that the Tits algebras are completely split (and the same for the Rost invariant).

If G is a group of inner type A or C with trivial Tits algebras, then G is split and the statement follows. If G is a group of type B or D, then the statement follows from Proposition 6.14 (In Prop. 6.14 we assume that the quadratic form q is even-dimensional. We use it only to conclude that $q \in I$ as a starting point in the proof.)

Let now G be a group of an exceptional type. Assume that the $K(2)$ -motive of X is split, but the Rost invariant of G is not trivial. By [PS10, Thm. 5.7] if G is not split already, there is a field extension K of F such that the Rost invariant of G_K is a pure non-zero symbol (For example, if $p = 2$ and G is of type E_8 , then one can take $K = F(Y)$ with Y the variety of maximal parabolic subgroups of G of type 6; enumeration of simple roots follows Bourbaki). Then the motive of X is a direct sum of Rost motives corresponding to this symbol of degree 3 (see [PSZ08]). This gives a contradiction with Prop. 6.4.

Conversely, if the Rost invariant of G is zero and G is not of type E_8 , then by [Ga01, Thm. 0.5] the group G is split and the statement follows. If G is of type E_8 with trivial Rost invariant, then by [Sem13, Theorem. 8.7] G has an invariant $u \in H_{\text{et}}^5(F, \mathbb{Z}/2)$ such that for a field extension K/F the invariant $u_K = 0$ iff G_K splits over a field extension of K of odd degree. Exactly as in the proof of Prop. 6.14 we can reduce to the case when u is a pure symbol. But then by [PSZ08] the motive of the variety X (Chow motive and cobordism motive and hence Morava-motive) is a direct sum of Rost motives for u and by Prop. 6.4 the $K(2)$ -Rost motive for a symbol of degree $5 > 3$ is split.

(4) If G is split by an odd degree field extension, then the $K(m)$ -motives of X are split for all m , since $p = 2$. Conversely, if G does not split over an odd degree field extension of F , then the invariant u is not zero. By [OVV07, Thm. 2.10] there is field extension K of F such that u_K is a non-zero pure symbol. Over K the motive of X is a direct sum of Rost motives corresponding to u_K . By Prop. 6.4 the $K(m)$ -Rost motives for a symbol of degree 5 are not split, if $m \geq 4$. \square

Finally we remark that sequence (4.5) can be used to define the Rost invariant in general, the invariant f_5 for groups of type F_4 and an invariant of degree 5 for groups of type E_8 with trivial Rost invariant (see [Sem13]). Namely, for the Rost invariant let G be a simple simply-connected algebraic group over F . Let Y be a G -torsor and set $n = 3$. Then sequence (4.5) gives an exact sequence

$$0 \rightarrow H_{\mathcal{M}}^{3,2}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Ker} \left(H_{\text{et}}^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H_{\text{et}}^3(F(Y), \mathbb{Q}/\mathbb{Z}(2)) \right) \rightarrow 0$$

But by sequence (5.1) $\text{Ker} \left(H_{\text{et}}^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H_{\text{et}}^3(F(Y), \mathbb{Q}/\mathbb{Z}(2)) \right)$ is a finite cyclic group. Therefore $H_{\mathcal{M}}^{3,2}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ is a finite cyclic group and the Rost invariant of Y is the image of $1 \in H_{\mathcal{M}}^{3,2}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ in $H_{\text{et}}^3(F, \mathbb{Q}/\mathbb{Z}(2))$.

To construct invariants of degree 5 for F_4 (resp. for E_8) one takes $n = 5$ and Y to be the variety of parabolic subgroups of type 4 for F_4 (the enumeration of simple roots follows Bourbaki) and resp. the variety of parabolic subgroups of any type for E_8 . In both cases $H_{\mathcal{M}}^{5,4}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ is cyclic of order 2 and the invariant is the image of $1 \in H_{\mathcal{M}}^{5,4}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ in $H_{\text{et}}^5(F, \mathbb{Q}/\mathbb{Z}(4))$; see [Sem13].

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